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Weak Copositive and Intertwining Approximation

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It is known that shape preserving approximation has lower rates than unconstrained approximation. This is especially true for copositive and intertwining approximations. For $f \in \mathbf{L}_p$, $1 \leqslant p < \infty$, the former only has rate $\omega_\varphi(f, n^{-1})_p$, and the latter cannot even be bounded by $C \|f\|_p$. In this paper, we discuss various ways to relax the restrictions in these approximations and conclude that the most sensible way is the so-called *almost* copositive/intertwining approximation in which one relaxes the restriction on the approximants in a neighborhood of radius $\Delta_n(y_j)$ of each sign change y_j.

Key Words: constrained approximation; almost copositive approximation; almost intertwining approximation; polynomials; splines; degree of approximation; Sobolev space.

1. INTRODUCTION

Let $\mathbf{C}[a, b]$ and $\mathbf{C}^k[a, b]$ be the sets of all continuous and all k-times continuously differentiable functions on $[a, b]$, respectively, and let $\mathbf{L}_p[a, b]$, $0 < p < \infty$, be the set of measurable functions on $[a, b]$ such that $\|f\|_{\mathbf{L}_p[a, b]} < \infty$, where

$$\|f\|_{\mathbf{L}_p[a, b]} := \left\{ \int_a^b |f(x)|^p \, dx \right\}^{1/p}.$$

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Throughout this paper $\mathbf{L}_\infty[a,b]$ is understood as $\mathbf{C}[a,b]$ with the usual uniform norm, to simplify the notation, and $1 \leqslant p \leqslant \infty$ *is always assumed unless otherwise indicated.* We also denote by $\mathbf{W}_p^k[a,b]$ the Sobolev space, the set of all functions f on $[a,b]$ such that f^{k-1} are absolutely continuous and $f^{(k)} \in \mathbf{L}_p$, and by $\mathbf{P}_n$ the set of all polynomials of degree $\leqslant n$.

Let us recall some definitions of moduli of smoothness used throughout this paper. The mth symmetric difference of f is given by

$$\Delta_h^m(f, x, [a,b]) := \begin{cases} \sum_{i=0}^{m} \binom{m}{i} (-1)^{m-i} f\left(x - \frac{mh}{2} + ih\right), & \text{if} \quad x \pm \frac{mh}{2} \in [a,b], \\ 0, & \text{otherwise.} \end{cases}$$

Then the mth (usual) modulus of smoothness of $f \in \mathbf{L}_p[a,b]$ is defined by

$$\omega^m(f, t, [a,b])_p := \sup_{0 \leqslant h \leqslant t} \|\Delta_h^m(f, \cdot, [a,b])\|_{\mathbf{L}_p[a,b]}.$$

We shall also use the so-called τ-modulus, or Sendov–Popov modulus, an averaged modulus of smoothness, defined for all bounded measurable functions on $[a,b]$ by

$$\tau^m(f, t, [a,b])_p := \|\omega^m(f, \cdot, t)\|_{\mathbf{L}_p[a,b]},$$

where

$$\omega^m(f, x, t) := \sup\{|\Delta_h^m(f, y)| : y \pm mh/2 \in [x - mt/2, x + mt/2] \cap [a,b]\}$$

is the mth local modulus of smoothness of f. (We set $\tau^m(f, t, [a,b])_p := \infty$ if the function f is unbounded.) If the interval $[-1,1]$ is used in any of the above notations, it will be omitted for the sake of simplicity, for example,

$$\|f\|_p := \|f\|_{\mathbf{L}_p[-1,1]}, \qquad \omega^m(f,t)_p := \omega^m(f, t, [-1,1])_p.$$

The ω- and τ-moduli measure the smoothness of f over the interval uniformly. The "non-uniform" modulus that we use is the mth Ditzian–Totik modulus of smoothness, defined for $f \in \mathbf{L}_p[-1,1]$ by

$$\omega_\varphi^m(f,t)_p := \sup_{0 \leqslant h \leqslant t} \|\Delta_{h\varphi(\cdot)}^m(f, \cdot, [-1,1])\|_p,$$

with $\varphi(x) := \sqrt{1-x^2}$. Let $\Delta_n(x) := n^{-1}\sqrt{1-x^2} + n^{-2}$. The term of $\omega^m(f, \Delta_n(x))_\infty$ is also used in this paper.

Let $Y_s := \{y_1, ..., y_s : y_0 := -1 < y_1 < y_2 < \cdots < y_s < 1 =: y_{s+1}\}$, $s \geqslant 0$. We denote by $\Delta^0(Y_s)$ the set of all functions f such that $(-1)^{s-j} f(x) \geqslant 0$

for $x \in [y_j, y_{j+1}]$, $k=0, ..., s$, i.e., those that have $0 \leqslant s < \infty$ sign changes at the points in Y_s and are nonnegative near 1. In particular, $\Delta^0 := \Delta^0(Y_0)$ denotes the set of all nonnegative functions on $[-1, 1]$. Functions f and g which belong to the same class $\Delta^0(Y_s)$ are said to be *copositive* on $[-1, 1]$. *Copositive approximation* is the approximation of functions f from $\Delta^0(Y_s)$ class by polynomials or splines that are copositive with f. For $f \in \mathbf{L}_p[-1, 1]$ let

$$E_n(f)_p := \inf_{P_n \in \mathbf{P}_n} \|f - P_n\|_p$$

denote the degree of unconstrained approximation, and let

$$E_n^{(0)}(f, Y_s)_p := \inf_{P_n \in \mathbf{P}_n \cap \Delta^0(Y_s)} \|f - P_n\|_p$$

be the degree of copositive polynomial approximation of f. In particular,

$$E_n^{(0)}(f)_p := E_n^{(0)}(f, Y_0)_p := \inf_{P_n \in \mathbf{P}_n \cap \Delta^0} \|f - P_n\|_p$$

is the degree of *positive approximation*. The degree of *intertwining polynomial approximation* of functions $f \in \mathbf{L}_p[-1, 1]$ with respect to Y_s is given by

$$\tilde{E}_n(f, Y_s)_p := \inf\{\|P - Q\|_p : P, Q \in \mathbf{P}_n, P - f \in \Delta^0(Y_s) \text{ and } f - Q \in \Delta^0(Y_s)\}.$$

We call $\{P, Q\}$ an *intertwining pair* of polynomials for f with respect to Y_s if $P - f, f - Q \in \Delta^0(Y_s)$. In particular, when $s=0$, $\tilde{E}_n(f, Y_0)_p = \tilde{E}_n(f)_p$ is the degree of *one-sided polynomial approximation* of f.

While intertwining approximation was introduced by the authors [10] not long ago, positive, copositive, and one-sided approximations have been studied extensively in recent years.

Some main results are summarized in Tables I–III. (See [9–10] and the references therein.) From these tables we see the degrees are astonishingly low in $\mathbf{L}_p$, $p < \infty$. As an extreme, the degree of intertwining approximation is not even bounded by $\|f\|_p$ or $\tau(f, 1)_p$. Recently, Leviatian and Shevchuk [20] obtained higher degree of comonotone approximation in $\mathbf{C}[-1, 1]$ by relaxing the restriction in a neighborhood of radius $\Delta_n(y_j)$ of each sign change y_j. Inspired by their idea, we discuss in this paper various ways to relax the restrictions in copositive and intertwining approximations and conclude that the most sensible way is the so-called *almost* copositive/intertwining approximation, in which one gives up the "right" amount of restriction in change for higher degrees than those in Tables I–III. All these are defined in Section 2 and summarized in Section 3.

TABLE I

Positive Approximation

$p=\infty$	
$f \in \mathbf{C}$	$\exists P_n,\ P_n(x) \geq 0$, such that $\lvert f(x) - P_n(x) \rvert \leq C\omega^m(f, \Delta_n(x))_\infty$
$1 \leq p < \infty$	
$f \in \mathbf{L}_p$	$E_n^{(0)}(f)_p \leq C\tau^m(f, n^{-1})_p$
	$E_n^{(0)}(f)_p \leq C\omega_\varphi(f, n^{-1})_p$
	$E_n^{(0)}(f)_p \not\leq C\omega^2(f, 1)_p$
$f \in \mathbf{W}_p^1$	$E_n^{(0)}(f)_p \leq Cn^{-1}E_{n-1}(f')_p$

TABLE II

Copositive Approximation

$p=\infty$	
$f \in \mathbf{C}$	$E_n^{(0)}(f, Y_s)_\infty \leq C\omega_\varphi^3(f, n^{-1})_\infty$
	$\exists P_n$, copositive with f, such that $\lvert f(x) - P_n(x) \rvert \leq C\omega^3(f, \Delta_n(x))_\infty$
	$E_n^{(0)}(f, Y_s)_\infty \not\leq C\omega^4(f, n^{-1})_\infty$
$f \in \mathbf{C}^1$	$E_n^{(0)}(f, Y_s)_\infty \leq Cn^{-1}\omega_\varphi^m(f', n^{-1})_\infty$
	$\exists\ P_n$, copositive with f, such that $\lvert f(x) - P_n(x) \rvert \leq C\Delta_n(x)\omega^m(f', \Delta_n(x))_\infty$
$1 \leq p < \infty$	
$f \in \mathbf{L}_p$	$E_n^{(0)}(f, Y_s)_p \leq C\tau^3(f, n^{-1})_p$
	$E_n^{(0)}(f, Y_s)_p \leq C\omega_\varphi(f, n^{-1})_p$
	$E_n^{(0)}(f, Y_s)_p \not\leq C\omega^2(f, 1)_p$
	$E_n^{(0)}(f, Y_s)_p \not\leq C\tau^4(f, 1)_p$
$f \in \mathbf{W}_p^1$	$E_n^{(0)}(f, Y_s)_p \leq Cn^{-1}\omega_\varphi^2(f', n^{-1})_p$
	$E_n^{(0)}(f, Y_s)_p \leq Cn^{-1}\tau^m(f', n^{-1})_p$
	$E_n^{(0)}(f, Y_s)_p \not\leq C\omega^3(f', 1)_p$
$f \in \mathbf{W}_p^2$	$E_n^{(0)}(f, Y_s)_p \leq Cn^{-2}\omega_\varphi^m(f'', n^{-1})_p$

TABLE III

Intertwining Approximation

$p=\infty$	
$f \in \mathbf{C}$	$\widetilde{E}_n(f, Y_s)_\infty \not\leq C\lVert f \rVert_\infty$
$f \in \mathbf{C}^1$	$\widetilde{E}_n(f, Y_s)_\infty \leq Cn^{-1}\omega_\varphi^m(f', n^{-1})_\infty$
	$\exists$ an intertwining pair $\{P_n, Q_n\}$ for f satisfying $\lvert P_n(x) - Q_n(x) \rvert \leq C\Delta_n(x)\omega^m(f', \Delta_n(x))_\infty$
$1 \leq p < \infty$	
$f \in \mathbf{L}_p$	$\widetilde{E}_n(f, Y_s)_p \not\leq C\lVert f \rVert_p$
	$\widetilde{E}_n(f, Y_s)_p \not\leq C\tau(f, 1)_p$
$f \in \mathbf{W}_p^1$	$\widetilde{E}_n(f, Y_s)_p \not\leq C\lVert f' \rVert_p$
	$\widetilde{E}_n(f, Y_s)_p \leq Cn^{-1}\tau^m(f', n^{-1})_p$
$f \in \mathbf{W}_p^2$	$\widetilde{E}_n(f, Y_s)_p \leq Cn^{-2}\omega_\varphi^m(f'', n^{-1})_p$

2. NOTATIONS AND DEFINITIONS

We denote $J_j(n, \varepsilon) := [y_j - \Delta_n(y_j)\, n^\varepsilon, y_j + \Delta_n(y_j)\, n^\varepsilon] \cap [-1, 1], j = 0, 1, ..., s+1$, and denote $O_n(Y_s, \varepsilon) := \bigcup_{j=1}^{s} J_j(n, \varepsilon)$ and $O_n^*(Y_s, \varepsilon) := \bigcup_{j=0}^{s+1} J_j(n, \varepsilon)$. If $\varepsilon = 0$, we shall also use the simpler notation $J_j := J_j(n, 0)$, $O_n(Y_s) := O_n(Y_s, 0)$, and $O_n^*(Y_s) := O_n^*(Y_s, 0)$. Functions f and g are said to be copositive on $J \subset I := [-1, 1]$ if $f(x)\, g(x) \geqslant 0$, $\forall x \in J$. Functions f and g are called *almost copositive* on I with respect to Y_s if they are copositive on $I \backslash O_n^*(Y_s)$. We say that f and g are *strongly* (*weakly*) *almost copositive* on I with respect to Y_s if they are copositive on $I \backslash O_n^*(Y_s, \varepsilon)$, where $\varepsilon < 0$ ($\varepsilon > 0$). In particular, if $\varepsilon = -\infty$, then strongly almost copositive functions are just copositive. We define a function class

$$(\varepsilon\text{-alm } \Delta)_n^0 (Y_s) := \{ f : (-1)^{s-k} f(x) \geqslant 0 \text{ for } x \in I \backslash O_n^*(Y_s, \varepsilon) \}.$$

If $s = 0$, it becomes

$$\begin{aligned}(\varepsilon\text{-alm } \Delta)_n^0 &:= (\varepsilon\text{-alm } \Delta)_n^0 (Y_0)\\ &:= \{ f : f(x) \geqslant 0 \text{ for } x \in [-1 + n^{-2+\varepsilon}, 1 - n^{-2+\varepsilon}] \},\end{aligned}$$

the set of all *strongly* (*weakly*) *almost nonnegative* functions on I if $\varepsilon < 0$ ($\varepsilon > 0$). Again, if $\varepsilon = 0$, we omit the letter ε in the notation and use $(\text{alm } \Delta)_n^0 (Y_s)$ and $(\text{alm } \Delta)_n^0$. The latter is the set of *almost nonnegative* function on I. If $\varepsilon = -\infty$, strongly almost nonnegative functions are just nonnegative.

DEFINITION. The degree of *almost positive* polynomial approximation of $f \in \mathbf{L}_p[-1, 1]$ is

$$E_n^{(0)}(f, \text{alm } Y_0)_p := \inf \{ \|f - P\|_p : P \in \mathbf{P}_n \cap (\text{alm } \Delta)_n^0 \}.$$

Similarly, we define $E_n^{(0)}(f, \varepsilon\text{-alm } Y_0)_p$, the degree of *strongly* (*weakly*) *almost positive* polynomial approximation of $f \in \mathbf{L}_p[-1, 1]$, by means of $P \in \mathbf{P}_n \cap (\varepsilon\text{-alm } \Delta)_n^0$.

DEFINITION. The degree of *almost copositive* polynomial approximation of $f \in \mathbf{L}_p[-1, 1] \cap \Delta^0(Y_s)$ is

$$E_n^{(0)}(f, \text{alm } Y_s)_p := \inf \{ \|f - P\|_p : P \in \mathbf{P}_n \cap (\text{alm } \Delta)_n^0 (Y_s) \}.$$

Similarly, we define $E_n^{(0)}(f, \varepsilon\text{-alm } Y_s)_p$, the degree of *strongly* (*weakly*) *almost copositive* polynomial approximation of $f \in \mathbf{L}_p[-1, 1] \cap \Delta^0(Y_s)$, by means of $P \in \mathbf{P}_n \cap (\varepsilon\text{-alm } \Delta)_n^0 (Y_s)$.

DEFINITION. The degree of *almost intertwining* polynomial approximation of $f \in \mathbf{L}_p[-1, 1]$ with respect to Y_s is

$$\begin{aligned}
&\tilde{E}_n(f, \text{alm } Y_s)_p \\
&\quad := \inf\{\|P - f\|_p + \|f - Q\|_p : P, Q \in \mathbf{P}_n, (-1)^{s-j}(P(x) - f(x)) \geqslant 0 \\
&\qquad\qquad \text{and } (-1)^{s-j}(f(x) - Q(x)) \geqslant 0 \\
&\qquad\qquad \text{if } x \in [y_j, y_{j+1}] \setminus O_n(Y_s), j = 0, ..., s\}.
\end{aligned}$$

We call $\{P, Q\}$ an almost intertwining pair of polynomials for f with respect to Y_s if P and Q satisfy the restrictions in the above infimum.

Note. We do not use $\|P - Q\|_p$ in the definition since $f(x)$ does not have to be between $P(x)$ and $Q(x)$ when x is close to y_j.

DEFINITION. The degree of *nearly intertwining* polynomial approximation of $f \in \mathbf{L}_p[-1, 1]$ with respect to Y_s is

$$\begin{aligned}
&\tilde{E}_n(f, \text{nearly } Y_s)_p \\
&\quad := \{\|P - Q\|_p : P, Q \in \mathbf{P}_n, P - f \in \Delta^0(\tilde{Y}_s) \text{ and } f - Q \in \Delta^0(\tilde{Y}_s), \\
&\qquad \text{where } \tilde{Y}_s = \{\tilde{y}_1, ..., \tilde{y}_s : \tilde{y}_0 := -1 < \tilde{y}_1 < \cdots < \tilde{y}_s < 1 := \tilde{y}_{s+1}\}, \\
&\qquad \text{and } |\tilde{y}_j - y_j| \leqslant \Delta_n(y_j) \text{ for } j = 1, 2, ..., s\}.
\end{aligned}$$

We call $\{P, Q\}$ a nearly intertwining pair of polynomials for f with respect to Y_s if $P - f, f - Q \in \Delta^0(\tilde{Y}_s)$.

Remark. We have the following relationships among the above quantities:

$$\tilde{E}_n(f, \text{alm } Y_s)_p \leqslant \tilde{E}_n(f, \text{nearly } Y_s)_p \leqslant \tilde{E}_n(f, Y_s)_p;$$

and for $f \in \Delta^0(Y_s)$,

$$E_n^{(0)}(f, \text{alm } Y_s)_p \leqslant \tilde{E}_n(f, \text{alm } Y_s)_p$$

and

$$E_n^{(0)}(f, \text{alm } Y_s)_p \leqslant E_n^{(0)}(f, Y_s)_p.$$

3. MAIN RESULTS

We summarize all the results in this paper in Tables IV–VII. Compared with Tables I–III we see significant improvements made by switching from positive/copositive/intertwining approximations to *almost* positive/copositive/intertwining approximations, due to relaxing restrictions in a neighborhood of radius $\Delta_n(y_j)$ of each y_j. For example, almost positive approximation improves to the order of $\omega_\varphi^2(f, n^{-1})_p$ from $\omega_\varphi(f, n^{-1})_p$ for $1 \leqslant p < \infty$. Almost intertwining approximation also achieves a better order of $\omega_\varphi^m(f, n^{-1})_\infty$ or $\omega^m(f, \Delta_n(x))_\infty$ in $\mathbf{C}$, and the order of $\tau^m(f, n^{-1})_p$ in $\mathbf{L}_p$, $1 \leqslant p < \infty$, although it still fails to reach the order of $\|f\|_p$ for $1 \leqslant p < \infty$, (for example, $f(x) := 0$, if $x \neq 0$, and $f(0) := 1$, and 0 is "far from" the set Y_s). Nearly intertwining approximation, in which the intertwining points are allowed to shift by an amount no larger than $\Delta_n(y_j)$ (using $\tilde{Y}_s$ instead of Y_s), improves to the order of $n^{-1}\omega_\varphi^m(f', n^{-1})_p$ from $n^{-1}\tau^m(f', n^{-1})_p$ for $f \in \mathbf{W}_p^1$. Unfortunately, it shows no improvements in $\mathbf{L}_p$. We emphasize that all rates we obtain in this paper are exact in the sense that one can not raise the order of the modulus used in the upper bound.

At the same time, we find that strongly almost positive/copositive approximations, in which restrictions are relaxed in intervals smaller than $[y_j - \Delta_n(y_j), y_j + \Delta_n(y_j)]$, do not do better than the ordinary positive/copositive approximations; while weakly almost positive/copositive approximations, in which restrictions are relaxed in intervals larger than $[y_j - \Delta_n(y_j), y_j + \Delta_n(y_j)]$, fail to bring a further improvement to the approximation order. In this sense, the "almost" version ($\varepsilon = 0$) is the most sensible weak version.

TABLE IV

Weak Positive Approximation of $f \in \mathbf{L}_p$, $1 \leqslant p < \infty$

$\varepsilon = 0$: **Almost Positive Approximation**	
$E_n^{(0)}(f, \mathrm{alm}Y_0)_p \le C\omega_\varphi^2(f, n^{-1})_p$	Theorem 1
$E_n^{(0)}(f, \mathrm{alm}Y_0)_p \not\le C\omega^3(f, 1)_p$	Theorem 1
$\varepsilon < 0$: **Strongly Almost Positive Approximation**	
$E_n^{(0)}(f, \varepsilon\text{-alm}Y_0)_p \le C\omega_\varphi(f, n^{-1})_p$	follows from positive approximation
$E_n^{(0)}(f, \varepsilon\text{-alm}Y_0)_p \not\le C\omega^2(f, 1)_p$	Theorem 2
$0 < \varepsilon < 2$: **Weakly Almost Positive Approximation**	
$E_n^{(0)}(f, \varepsilon\text{-alm}Y_0)_p \le C\omega_\varphi^2(f, n^{-1})_p$	Theorem 1
$E_n^{(0)}(f, \varepsilon\text{-alm}Y_0)_p \not\le C\omega^3(f, 1)_p$	Theorem 1
$\varepsilon \ge 2$: **Weakly Almost Positive Approximation**	
Becomes unconstrained approximation	

Note. If $f \in \mathbf{C}$ or $\mathbf{W}_p^1$, then (almost) positive approximation has the same order as the unconstrained case (see Table I or [10]).

TABLE V

Weak Copositive Approximation

$\varepsilon = 0$: **Almost Copositive Approximation**		
$p = \infty$		
$f \in \mathbf{C}$	$E_n^{(0)}(f, \mathrm{alm}Y_s)_\infty \le C\omega_\varphi^m(f, n^{-1})_\infty$	Theorem 10
	$\exists\ P \in \mathbf{P}_n \cap (\mathrm{alm}\Delta)^0_n(Y_s)$ such that $\lvert f(x) - P(x)\rvert \le C\omega^m(f, \Delta_n(x))_\infty$	
$1 \le p < \infty$		
$f \in \mathbf{L}_p$	$E_n^{(0)}(f, \mathrm{alm}Y_s)_p \le C\tau^m(f, n^{-1})_p$	Theorem 11
	$E_n^{(0)}(f, \mathrm{alm}Y_s)_p \le C\omega_\varphi^2(f, n^{-1})_p$	Theorem 4
	$E_n^{(0)}(f, \mathrm{alm}Y_s)_p \not\le C\omega^3(f, n^{-1})_p$, $1 < p < \infty$ and $\not\le C\omega^4(f, n^{-1})_p$, if $p = 1$	Corollary 6
$f \in \mathbf{W}_p^1$	$E_n^{(0)}(f, \mathrm{alm}Y_s)_p \le Cn^{-1}\omega_\varphi^m(f', n^{-1})_p$	Theorem 11
$\varepsilon < 0$: **Strongly Almost Copositive Approximation**		
$1 \le p < \infty$ and $s = 1$		
$f \in \mathbf{L}_p$	$E_n^{(0)}(f, \varepsilon\text{-}\mathrm{alm}Y_s)_p \not\le C\omega^2(f, 1)_p$	Theorem 8
$1 \le p < \infty$ and $s > 1$		
$f \in \mathbf{L}_p$	$E_n^{(0)}(f, \varepsilon\text{-}\mathrm{alm}Y_s)_p \not\le C\omega^2(f, n^{-1})_p$, if $p > 1$	Theorem 9
	$E_n^{(0)}(f, \varepsilon\text{-}\mathrm{alm}Y_s)_p \not\le C\omega^3(f, n^{-1})_p$, if $p = 1$	
$0 < \varepsilon < 1$: **Weakly Almost Copositive Approximation**		
Same as almost copositive approximation for large n		
$\varepsilon > 1$: **Weakly Almost Copositive Approximation**		
Becomes unconstrained approximation for large n		

TABLE VI

Almost Intertwining Approximation

$p = \infty$		
$f \in \mathbf{C}$	$\widetilde{E}_n(f, \mathrm{alm}Y_s)_\infty \le C\omega_\varphi^m(f, n^{-1})_\infty$	Theorem 13
	$\exists$ an almost intertwining pair $\{P_n, Q_n\}$ such that $\lvert P_n(x) - f(x)\rvert + \lvert f(x) - Q_n(x)\rvert \le C\omega^m(f, \Delta_n(x))_\infty$	Theorem 13
$1 \le p < \infty$		
$f \in \mathbf{L}_p$	$\widetilde{E}_n(f, \mathrm{alm}Y_s)_p \le C\tau^m(f, n^{-1})_p$	Theorem 14
	$\widetilde{E}_n(f, \mathrm{alm}Y_s)_p \not\le C\lVert f\rVert_p$	Example on page 7
$f \in \mathbf{W}_p^1$	$\widetilde{E}_n(f, \mathrm{alm}Y_s)_p \le Cn^{-1}\omega_\varphi^m(f', n^{-1})_p$	Theorem 14

TABLE VII

Nearly Intertwining Approximation

$f \in \mathbf{L}_p$ $0 < p \le \infty$	$\widetilde{E}_n(f, \mathrm{nearly}Y_s)_p \not\le C\lVert f\rVert_p$	Theorem 15 and Example on page 7
	$\widetilde{E}_n(f, \mathrm{nearly}Y_s)_p \not\le C\tau(f, 1)_p$	
$f \in \mathbf{W}_p^1$	$\widetilde{E}_n(f, \mathrm{nearly}Y_s)_p \le Cn^{-1}\omega_\varphi^m(f', n^{-1})_p$	Theorem 17

In the next section, we discuss weak positive approximation. Sections 5 and 6 are devoted to weak copositive approximation and weak intertwining approximation, respectively.

4. WEAK POSITIVE APPROXIMATION

Although positive approximation is a special case of copositive approximation, it very often has a better rate; at least this is the case in the ordinary positive approximation. We begin with almost and weakly/strongly almost positive approximation for $1 \leqslant p < \infty$. Theorems 1 and 2 show that almost positive approximation has an order of $\omega_\varphi^2(f, n^{-1})_p$, compared with $\omega_\varphi(f, n^{-1})_p$ for the ordinary positive approximation. The rate is exact in the sense that one cannot replace it even by $\omega^3(f, 1)_p$. This is obtained by relaxing the restriction on intervals of length n^{-2} at $x = \pm 1$. Using larger intervals (of length $n^{-2+\varepsilon}$, $0 < \varepsilon < 2$) gains no more than this, unless giving up the restriction on the whole interval $[-1, 1]$ ($\varepsilon \geqslant 2$), that is, back to unconstrained approximation; while using smaller intervals ($\varepsilon < 0$) yields no improvement over the ordinary positive approximation. For the case of $p = \infty$, see the note below Table IV.

THEOREM 1. *Let $f \in \mathbf{L}_p[-1, 1] \cap \Delta^0$, $0 \leqslant \varepsilon < 2$, and $1 \leqslant p < \infty$. Then*

$$E_n^{(0)}(f, \varepsilon\text{-alm } Y_0)_p \leqslant C\omega_\varphi^2(f, n^{-1})_p, \tag{4.1}$$

where C is an absolute constant. On the other hand, given any $A > 0$, $n \in \mathcal{N}$, $1 \leqslant p < \infty$, and $0 \leqslant \varepsilon < 2$, $\exists f \in \mathbf{L}_p[-1, 1] \cap \Delta^0$ such that

$$E_n^{(0)}(f, \varepsilon\text{-alm } Y_0)_p > A\omega^3(f, 1)_p. \tag{4.2}$$

Proof. Inequality (4.1) follows from Theorem 4. To prove (4.2), we let $Q(x) := x^2 - b^{-1}$, where $b > 1$ is a constant to be chosen later, and define

$$f(x) := Q(x)_+ := \begin{cases} Q(x), & |x| > \sqrt{b^{-1}}, \\ 0, & \text{otherwise.} \end{cases}$$

Then

$$\begin{aligned}\omega^3(f, 1)_p &= \omega^3(f - Q, 1)_p \leqslant C\, \|f - Q\|_p \\ &\leqslant C\, \|Q\|_{\mathbf{L}_p[-\sqrt{b^{-1}}, \sqrt{b^{-1}}]} \leqslant Cb^{-1-1/(2p)}.\end{aligned}$$

Suppose, towards a contradiction, P_n is a polynomial from $\mathbf{P}_n$ such that $P_n(0) \geqslant 0$ and $\|P_n - f\|_p \leqslant A\omega^3(f, 1)_p$. Then

$$\begin{aligned}|P_n(0) - Q(0)| &\leqslant \|P_n - Q\|_\infty \leqslant C\,\|P_n - Q\|_p \leqslant C\,\|P_n - f\|_p + C\,\|f - Q\|_p \\ &\leqslant CA\omega^3(f, 1)_p + C\,\|f - Q\|_p \leqslant C_1 b^{-1-1/(2p)},\end{aligned}$$

where C_1 depends on n but not on b. Therefore,

$$\begin{aligned}P_n(0) &\leqslant Q(0) + C_1 b^{-1-1/(2p)} = -b^{-1} + C_1 b^{-1-1/(2p)} \\ &= b^{-1-1(2p)}(C_1 - b^{1/(2p)}) < 0\end{aligned}$$

for sufficiently large b, which is the desired contradiction. ∎

Remark. The same proof can be used to show that for any $\beta > 0$ and $0 \leqslant \varepsilon < 2$,

$$E_n^{(0\text{-})}(f, \varepsilon\text{-alm } Y_0)_p \nleqslant Cn^\beta \omega^3(f, 1)_p, \qquad 1 \leqslant p < \infty.$$

THEOREM 2. *For any given $A > 0$, $1 \leqslant p < \infty$, $\varepsilon < 0$, and sufficiently large n, $\exists f \in \mathbf{L}_p[-1, 1] \cap \Delta^0$ such that*

$$E_n^{(0)}(f, \varepsilon\text{-alm } Y_0)_p > A\omega^2(f, 1)_p. \tag{4.3}$$

Proof. We only give a sketch since the proof is similar to that of Theorem 1. Let $L(x) := x + 1 - n^{-2}a$, and $f(x) := L(x)_+ := \max\{L(x), 0\}$. By choosing the value of the parameter $a > 4n^\varepsilon$ carefully one can readily prove that

$$P_n(-1 + 2n^{\varepsilon - 2}) < 0$$

for any polynomial $P_n \in \mathbf{P}_n$ with $\|P_n - f\|_p \leqslant A\omega^2(f, 1)_p$ provided n is sufficiently large. Therefore P_n is not strongly almost positive. ∎

5. WEAK COPOSITIVE APPROXIMATION

In this section, we first show in Theorem 4 that almost copositive approximation in $\mathbf{L}_p$, $1 \leqslant p < \infty$, improves the rate to $\omega_\varphi^2(f, n^{-1})_p$ from $\omega_\varphi(f, n^{-1})_p$, the rate for the ordinary copositive approximation. We first need an analog of this for splines. The result for polynomials will then come from the following theorem by the authors [10].

THEOREM A. *Let Y_s $(s \geqslant 0)$ be given, $m \in \mathcal{N}$, $\mu \geqslant 2m + 30$, $0 < p \leqslant \infty$, and let $S(x)$ be a spline of an odd order r $(r = 2m + 1)$ on the knot sequence $\{x_i = \cos(i\pi/n)\}_{i \in I_n(Y_s)}$, where $n > C(Y_s)$ is such that there are at least 4* *knots* x_i *in each interval* (y_j, y_{j+1}), $j=0, ..., s$, *and* $I_n(Y_s) := \{1, ..., n\} \setminus \{i, i-1: x_i \leqslant y_j < x_{i-1}$ *for some* $1 \leqslant j \leqslant s\}$. *Then there exists an intertwining pair of polynomials* $\{P_1, P_2\} \subset \mathbf{P}_{C(r)\,n}$ *for* S *with respect to* Y_s *such that*

$$\|P_1 - P_2\|_p^p \leqslant C(r, s, \min\{p, 1\})^p \sum_{i=1}^{n-1} E_{r-1}(S, \hat{I}_i \cup \hat{I}_{i+1})_p^p, \qquad \text{if} \quad 0 < p < \infty, \tag{5.1}$$

and

$$|P_1(x) - P_2(x)| \leqslant C(r, \mu, s) \sum_{i=1}^{n-1} E_{r-1}(S, \hat{I}_i \cup \hat{I}_{i+1})_\infty \left(\frac{|\hat{I}_i|}{|x - x_i| + |\hat{I}_i|} \right)^\mu, \qquad \text{if} \quad p = \infty, \tag{5.2}$$

where $\hat{I}_i := [x_i, x_{i-1}]$ *and* $E_n(f, [a, b])_p := \inf_{P_n \in \mathbf{P}_n} \|f - P_n\|_{\mathbf{L}_p[a, b]}$.

Let $k > 0$ be an integer, and $-1 = t_0 < t_1 < \cdots < t_{k-1} < t_k = 1$ be a partition of $I = [-1, 1]$. Define the so-called auxiliary knots (DeVore and Lorentz [5, p. 140]) by $t_i := -1 + i\Delta t_0$, $i = -r+1, ..., -1$, and $t_i := 1 + (i-k)\,\Delta t_{k-1}$, $i = k+1, ..., k+r-1$, where $\Delta t_i := t_{i+1} - t_i$. Let $I_i := [t_i, t_{i+1}]$, and $J_i := [t_{i-r+1}, t_{i+r}]$. Denote $\mathbf{T}_k := \{t_i\}_{i=-r+1}^{k+r-1}$, $\Delta\mathbf{T}_k := \max\{\Delta t_i\}$, and for $i = -r+1, ..., k-1$, $d_i := (t_{i+r} - t_i)/r$, $t_i^* := (t_{i+1} + \cdots + t_{i+r-1})/(r-1)$, $d_i^* := 2\min(t_i^* - t_i, t_{i+r} - t_i^*)/r$, and $I_i^* := [t_i^* - d_i^*/2, t_i^* + d_i^*/2]$. For any $f \in \mathbf{L}_1[t_{-r+1}, t_{k+r-1}]$, we define a linear operator T by

$$Tf := \sum_{i=-r+1}^{k-1} c_i N_i, \qquad c_i := c_i(f) := d_i^{*\,-1} \int_{I_i^*} f,$$

where $N_i(x) := N_{r,i}(x) := N(x; t_i, ..., t_{i+r})$ is the B-spline on $t_i, ..., t_{i+r}$ normalized so that $\sum N_i(x) \equiv 1$. Note that T preserves linearity, that is, $Tl = l$ for any $l \in \mathbf{P}_1$, because T becomes the Schoenberg variation diminishing operator in this case [2, Chap. XI–XII].

For any function $f \in \mathbf{L}_p[-1, 1]$, we use its Whitney's Extension to $[t_{-r+1}, t_{k+r-1}]$ so that T can be applied. This will only enlarge the constant C in (5.3) and (5.4) by a factor depending only on r (see Theorem 6.4.1 and its proof in [5]). And we shall still use the letter f for the extension for the sake of simple notation. With the notation above, we prove

LEMMA 3. *Let* $f \in \mathbf{L}_p[-1, 1]$. *Then the spline* $S := Tf$ *of order* r *on the knot sequence* $\mathbf{T}_k$ *satisfies*

$$\|f-S\|_p \leqslant C\omega^2(f, \Delta\mathbf{T}_k)_p, \tag{5.3}$$

$$\|f-S\|_p^p \leqslant C^p \sum_{i=-r+1}^{k-1} \omega^2(f, \Delta t_i, J_i)_p^p, \tag{5.4}$$

where the constant C depends on r and the ratios $\Delta t_i/\Delta t_{i-1}$ of lengths of neighboring subintervals.

Proof. By Hölder's Inequality we have

$$|c_i| \leqslant d_i^{*\,-1} \int_{I_i^*} |f| \leqslant d_i^{*\,-1} \|f\|_{\mathbf{L}_p(I_i^*)} d_i^*\, 1-\frac{1}{p} = d_i^{*\,-1/p} \|f\|_{\mathbf{L}_p(I_i^*)}.$$

By the well-known relationship between a spline and its B-spline series coefficients (de Boor, see [2, Chap. XI] and [21, Section 4.6] for $p=\infty$, and [5, Section 5.4] for $0<p<\infty$)

$$\begin{aligned}\|Tf\|_{\mathbf{L}_p(I_i)} = \|S\|_{\mathbf{L}_p(I_i)} &= \left\| \sum_{j=i-r+1}^{i} c_j N_j \right\|_{\mathbf{L}_p(I_i)} \leqslant C \sum_{j=i-r+1}^{i} |c_j|\, d_j^{1/p} \\ &\leqslant C \sum_{j=i-r+1}^{i} \|f\|_{\mathbf{L}_p(I_j^*)} (d_j/d_j^*)^{1/p} \leqslant C\, \|f\|_{\mathbf{L}_p(J_i)}.\end{aligned} \tag{5.5}$$

Let l_i be a best linear approximation to f on I_i; then

$$\|f-l_i\|_{\mathbf{L}_p(I_i)} \leqslant C\omega^2(f, \Delta t_i, I_i)_p.$$

This l_i is also a near best linear approximation on $J_i = [t_{i-r+1}, t_{i+r}] \supseteq I_i$ (DeVore and Popov [6]) and therefore satisfies

$$\|f-l_i\|_{\mathbf{L}_p(J_i)} \leqslant C\omega^2(f, \Delta t_i, J_i)_p.$$

Applying (5.5) to $T(f-l_i)$ gives (5.4);

$$\begin{aligned}\|f-S\|_{\mathbf{L}_p(I_i)} &= \|f-Tf\|_{\mathbf{L}_p(I_i)} \leqslant \|f-l_i\|_{\mathbf{L}_p(I_i)} + \|T(f-l_i)\|_{\mathbf{L}_p(I_i)} \\ &\leqslant C(\omega^2(f, \Delta t_i, I_i)_p + \|f-l_i\|_{\mathbf{L}_p(J_i)}) \leqslant C\omega^2(f, \Delta t_i, J_i)_p,\end{aligned}$$

which in turn gives (5.3) (Leviatan and Mhaskar [19], also see Hu [8]):

$$\|f-S\|_p \leqslant C\omega^2(f, \Delta\mathbf{T}_k, [t_{-r+1}, t_{k+r-1}])_p \leqslant C\omega^2(f, \Delta\mathbf{T}_k, [-1, 1])_p. \quad \blacksquare$$

The following theorem gives affirmative results on almost copositive approximation. Here and throughout the rest of the paper, we denote $J_j^- := [y_j - \Delta_n(y_j), y_j]$ and $J_j^+ := [y_j, y_j + \Delta_n(y_j)]$.

THEOREM 4. *Suppose* $f \in \mathbf{L}_p[-1,1] \cap \Delta^0(Y_s)$.

(i) *Let* $\mathbf{T}_k$ *be a given single-knot sequence such that there are at least* r *knots in each of* J_j^-, $j=1, ..., s+1$, *and in each of* J_j^+, $j=0, ..., s$, *if they do not intersect* J_{j-1}^+ *or* J_{j+1}^-, *respectively. Then* $S := Tf = \sum_{i=-r+1}^{k-1} c_i N_i$ *is almost copositive with* f.

(ii) *For any* $n > C(Y_s)$, *there exists a polynomial* $P \in \mathbf{P}_n$ *that is almost copositive with* f *and satisfies*

$$\|f-P\|_p \leqslant C(s)\, \omega_\varphi^2(f, n^{-1})_p. \tag{5.6}$$

Remark. Although we require $n > C(Y_s)$ in (ii) for simplicity, it seems unnecessary. In many cases of constrained approximation how large n is depends on Y_s because two y_j's may be very close to each other, and the degree of a polynomial will then have to be very large to follow the trend of the graph. In this paper, however, we relax the shape-preserving requirement in a neighborhood of each y_j of radius $\Delta_n(y_j)$. When some points in Y_s get too close to one another, these neighborhoods will be connected and we will not have to worry about the sign changes at these points. In other words, the set Y_s can be "thinned out" if its points are dense (or, equivalently, if n is small).

Proof. For (i), we only prove that S is copositive with f between J_0^+ and J_1^-, which is $A_0 := [-1+n^{-2}, y_1 - \Delta_n(y_1)]$, if the two intervals do not intersect. For the sake of certainty, we assume f is nonnegative on A_0. Since there are at least r knots in J_0^+, if we denote the last knot less than $y_1 - \Delta_n(y_1)$ by t_{j_1}, then the restriction of S on A_0 can be written as $S|_{A_0} = \sum_{i=0}^{j_1} c_i N_i$. From the facts that c_i is the integral average of f on $I_i^* \subset [t_i, t_{i+r}]$, and that there are at least r knots in each of J_0^+ and J_1^-, we know $c_i \geqslant 0$, $i=0, ..., j_1$. Thus $S|_{A_0} \geqslant 0$.

To prove (ii), we take $r=3$ in Lemma 3 and choose $k \geqslant C_1 n$ large enough so that the knot sequence $\mathbf{T}_{k-2s} := \{x_i = \cos(i\pi/k)\}_{i \in I_k(Y_s)}$ has at least 4 knots in each of J_j^-, $j=1, ..., s+1$, and J_j^+, $j=0, ..., s$, if they do not intersect J_{j-1}^+ or J_{j+1}^-, respectively. We add auxiliary knots to $\mathbf{T}_{k-2s}$ as before Lemma 3, and define a quadratic spline on $\mathbf{T}_{k-2s}$ by $S := Tf$. By Lemma 3 and Theorem A,

$$\begin{aligned}\|S-P\|_p^p &\leqslant C^p \sum_{i=1}^{k-1} E_2(S, \hat{I}_i \cup \hat{I}_{i+1})_p^p \leqslant C^p \sum_{i=1}^{k-1} \omega^2(f\,|\hat{I}_i \cup \hat{I}_{i+1}|, \hat{I}_i \cup \hat{I}_{i+1})_p^p \\ &\leqslant C^p \omega_\varphi^2(f, n^{-1})_p^p,\end{aligned}$$

where in the last step we have used an inequality established and/or used in [3, 4, 9, 16, 10]. The fact that P is almost copositive with f follows from (i) and Theorem A. ∎

In the following four theorems and corollary, we show that (5.3), (5.4), and (5.6) are exact for $1 < p < \infty$ in the sense that one can not replace ω^2 or ω_φ^2 by ω^3 in them (Corollary 6); that weakly almost copositive approximation does not do better than these (Corollary 6), and strongly almost copositive approximation does not do better than the ordinary copositive approximation (Theorems 8 and 9), in spite of larger intervals in which the restriction is relaxed.

THEOREM 5. *Let Y_s be fixed. For any given $A > 0$, $1 \leqslant p < \infty$, and sufficiently large $n \in \mathcal{N}$, there exists a function $f \in \mathbf{C}[-1, 1] \cap \Delta^0(Y_s)$ such that for every polynomial $P_n \in \mathbf{P}_n$, which is copositive with f on $[y_s + (1 - y_s)/3, 1 - (1 - y_s)/3]$, the following inequality holds,*

$$\|f - P_n\|_p > A n^\beta \omega^m(f, n^{-1})_p, \tag{5.7}$$

where $m = 3$ and $\beta < (p-1)/p(2p+1)$ if $1 < p < \infty$, and $m = 4$ and $\beta < 1/3$ if $p = 1$.

COROLLARY 6. *Let Y_s be fixed. For any given $0 \leqslant \varepsilon < 1$, $A > 0$, $1 \leqslant p < \infty$, and sufficiently large $n \in \mathcal{N}$, there exists $f \in \mathbf{C}[-1, 1] \cap \Delta^0(Y_s)$ such that*

$$E_n^{(0)}(f, \varepsilon\text{-alm } Y_s)_p > A \begin{cases} \omega^3(f, n^{-1})_p, & \text{if } \ 1 < p < \infty \\ \omega^4(f, n^{-1})_p, & \text{if } \ p = 1. \end{cases}$$

To prove Theorem 5, we need the following inequality for polynomials. We were aware of its usage in Zhou [23] through communication with him, but could not find a handy reference. The following is a modification of the proof Professor Zhou outlined to the authors.

LEMMA 7. *Let $P_n \in \mathbf{P}_n$ and $1 \leqslant p < \infty$. Then*

$$|P_n(x)\sqrt{1 - x^2}| \leqslant C_p n^{1/p} \|P_n\|_p, \qquad x \in [-1, 1].$$

Proof. Let $x := \cos\theta$ and $t_n(\theta) := P_n(\cos\theta)\sin\theta$. Since t_n is a trigonometric function of degree $n + 1$, applying the Nikolskii's inequality, we have

$$|t_n(\theta)| \leqslant C_p n^{1/p} \|t_n\|_{\mathbf{L}_p[0, 2\pi]}.$$

Notice that

$$\|t_n\|^p_{\mathbf{L}_p[0,2\pi]} = \int_0^{2\pi} |t_n(\theta)|^p\, d\theta = 2\int_0^{\pi} |P_n(\cos\theta)|^p\, (\sin\theta)^p\, d\theta$$

$$\leqslant 2\int_0^{\pi} |P_n(\cos\theta)|^p \sin\theta\, d\theta = 2\int_{-1}^{1} |P_n(x)|^p\, dx = 2\,\|P_n\|^p_p.$$

Therefore, we obtain

$$|P_n(x)\sqrt{1-x^{21}}| = |t_n(\theta)| \leqslant C_p n^{1/p}\,\|P_n\|_p. \quad \blacksquare$$

Proof of Theorem 5. The following is a modification of the proof used by Gilewicz and Shevchuk [7]. Let $n \geqslant s+2$, $x_0 := (1+y_s)/2$,

$$L(x) := (x - x_0 + b)(x - x_0 - b)\prod_{j=1}^{s}(x - y_j),$$

where $b < (1-y_s)/6$ is a constant to be chosen later, and let

$$f(x) := \begin{cases} L(x), & \text{if} \quad x \notin [x_0 - b, x_0 + b] \\ 0, & \text{otherwise.} \end{cases}$$

Suppose (5.7) is not true, i.e., there exists a polynomial $P_n \in \mathbf{P}_n$ such that $P_n(x) \geqslant 0$, $x \in [y_s + (1-y_s)/3, 1 - (1-y_s)/3]$ (therefore, $P_n(x) \geqslant 0$ for $x \in [x_0 - b, x_0 + b]$), and

$$\|f - P_n\|_p \leqslant A n^{\beta} \omega^m(f, n^{-1})_p.$$

Without loss of generality, we can assume $\beta \geqslant 0$. Note that

$$\|f - L\|_p = \left(\int_{x_0 - b}^{x_0 + b} |L(x)|^p\, dx\right)^{1/p} \leqslant C b^{2+1/p}$$

and

$$\omega^m(f, n^{-1})_p \leqslant \omega^m(f - L, n^{-1})_p + \omega^m(L, n^{-1})_p$$

$$\leqslant 2^m\,\|f - L\|_p + n^{-m}\,\|L^{(m)}\|_p \leqslant C b^{2+1/p} + C n^{-m}.$$

Also, by Lemma 7, we have

$$\|P_n - L\|_p$$

$$\geqslant C n^{-1/p}(P_n(x_0) - L(x_0))\sqrt{1 - x_0^2}$$

$$\geqslant -C(Y_s)\, n^{-1/p}\, L(x_0) = C(Y_s)\, n^{-1/p} b^2 \prod_{j=1}^{s}(x_0 - y_j) \geqslant C(Y_s)\, n^{-1/p} b^2.$$

Therefore,

$$C(Y_s)\, n^{-1/p} b^2 \leqslant \|P_n - L\|_p \leqslant \|P_n - f\|_p + \|f - L\|_p$$
$$\leqslant A n^{\beta} \omega^m(f, n^{-1})_p + C b^{2+1/p} \leqslant C n^{\beta} b^{2+1/p} + C n^{-m+\beta}.$$

This implies the inequality

$$b^2 (C(Y_s)\, n^{-1/p} - C n^{\beta} b^{1/p}) \leqslant C n^{-m+\beta}.$$

Now, let $b = c n^{-1-\beta p}$ with sufficiently small c, then the last inequality implies

$$n^{m-2-2\beta p-\beta-1/p} \leqslant C(p, Y_s).$$

But this cannot be true for sufficiently large n since the condition on m, β, and p in the theorem imply $m > 2 + 2\beta p + \beta + 1/p$. ▮

THEOREM 8. *For any given $A > 0$, $\varepsilon < 0$, $1 \leqslant p < \infty$, and sufficiently large $n \in \mathcal{N}$, there exists $f \in \mathbf{L}_p[-1, 1]$ that changes sign only once at $x = 0$ such that for every polynomial $P_n \in \mathbf{P}_n$ with $P_n(2n^{-1+\varepsilon}) \geqslant 0$ the following inequality holds*:

$$\|f - P_n\|_p > A\omega^2(f, 1)_p.$$

Proof. Here once again we omit details of the proof since the argument is similar to our previous counterexamples. Let $L(x) := n/a(x - a/n)$, and let

$$f(x) := \begin{cases} L(x), & \text{if } \; x \notin [0, a/n] \\ 0, & \text{otherwise.} \end{cases}$$

Then $f \in \mathbf{L}_p[-1, 1] \cap \Delta^0(Y_1)$, where $Y_1 = \{0\}$. By choosing the value of the parameter $a > 2n^{\varepsilon}$ carefully one can readily prove that $P_n(2n^{-1+\varepsilon}) < 0$ for any polynomial $P_n \in \mathbf{P}_n$ with $\|f - P_n\|_p \leqslant A\omega^2(f, 1)_p$ provided n is sufficiently large. ▮

THEOREM 9. *Let Y_s, $s \geqslant 1$, be fixed. For any given $\varepsilon < 0$, $A > 0$, $1 \leqslant p < \infty$, and sufficiently large $n \in \mathcal{N}$, there exists a function $f \in \mathbf{L}_p[-1, 1] \cap \Delta^0(Y_s)$ such that for every polynomial $P_n \in \mathbf{P}_n$ with $P_n(y_s + 2\Delta_n(y_s)\, n^{\varepsilon}) \geqslant 0$ the following inequality holds,*

$$\|f - P_n\|_p > A\omega^m(f, n^{-1})_p, \tag{5.8}$$

where $m = 2$ if $1 < p < \infty$, and $m = 3$ if $p = 1$. In particular, (5.8) holds for all polynomials that are strongly almost copositive with f.

Remark. As in Theorem 5, the inequality (5.8) can be improved to

$$\|f-P_n\|_p > An^{\beta}\omega^m(f,n^{-1})_p \qquad \text{for some} \quad \beta>0.$$

Proof. We use the same idea as in Theorem 5. Let $A>0$ be fixed, and let $n \geqslant 2^{-1/\varepsilon}$ and b be chosen later. Denote

$$L(x) := (x-y_s-b\Delta_n(y_s)) \prod_{j=1}^{s-1} (x-y_j)$$

and

$$f(x) := \begin{cases} L(x), & \text{if} \quad x \notin (y_s-b\Delta_n(y_s), y_s+b\Delta_n(y_s)), \\ 0, & \text{otherwise.} \end{cases}$$

Suppose that the assertion of the theorem is not true, i.e., there exists a polynomial $P_n \in \mathbf{P}_n$ such that $P_n(\tilde{x}) \geqslant 0$, where $\tilde{x} = y_s + 2\Delta_n(y_s)\, n^{\varepsilon}$, and

$$\|f-P_n\|_p \leqslant A\omega^m(f,n^{-1})_p.$$

Note that

$$\|f-L\|_p \leqslant \|L\|_{\mathbf{L}_p[y_s-b\Delta_n(y_s),\, y_s+b\Delta_n(y_s)]} \leqslant C(b\Delta_n(y_s))^{1+1/p} \leqslant C(Y_s)(b/n)^{1+1/p},$$

and

$$\begin{aligned} \omega^m(f,n^{-1})_p &\leqslant \omega^m(f-L,n^{-1})_p + \omega^m(L,n^{-1})_p \\ &\leqslant 2^m \|f-L\|_p + n^{-m} \|L^{(m)}\|_p \leqslant C(Y_s)(b/n)^{1+1/p} + Cn^{-m}. \end{aligned}$$

Now, by Lemma 7, we have

$$\begin{aligned} &P_n(\tilde{x}) - L(\tilde{x}) \\ &\quad\leqslant Cn^{1/p}(\sqrt{1-\tilde{x}^2})^{-1} \|P_n - L\|_p \\ &\quad\leqslant C(Y_s)\, n^{1/p}(\|P_n - f\|_p + \|f-L\|_p) \leqslant C(Y_s)\, n^{1/p}((b/n)^{1+1/p} + n^{-m}) \\ &\quad\leqslant C(Y_s)(b^{1+1/p}n^{-1} + n^{-m+1/p}). \end{aligned}$$

Since

$$\begin{aligned} L(\tilde{x}) &= -\Delta_n(y_s)(b-2n^{\varepsilon}) \prod_{j=1}^{s-1} (y_s - y_j + 2\Delta_n(y_s)\, n^{\varepsilon}) \\ &\leqslant -C_1(Y_s)\, n^{-1}(b-2n^{\varepsilon}), \end{aligned}$$

we have

$$P_n(\tilde{x}) \leqslant -C_1(Y_s)\, n^{-1}(b-2n^{\varepsilon}) + C(Y_s)(b^{1+1/p} n^{-1} + n^{-m+1/p}) < 0$$

if $b > 4n^{\varepsilon}$, $b^{1+1/p} > n^{-m+1+1/p}$ and $b < C^p$. This choice of b is possible if n is sufficiently large, and $m > 1 + 1/p$, which is true with the choices of m and p in the theorem. This is the desired contradiction. ∎

So far we have not mentioned anything about almost copositive approximation for $p = \infty$. The theorem below says it reaches the same rate as the unconstrained case. The result follows from Theorem 13, their analogue for almost intertwining approximation.

THEOREM 10. *Let $f \in C[-1, 1] \cap \Delta^0(Y_s)$, and m be a positive integer. Then*

$$E_n^{(0)}(f, \text{alm } Y_s)_\infty \leqslant C\omega_\varphi^m(f, n^{-1})_\infty.$$

Moreover, there exists a $P_n \in \mathbf{P}_n \cap (\text{alm } \Delta)_n^0(Y_s)$ such that

$$|f(x) - P_n(x)| \leqslant C\omega^m(f, \Delta_n(x))_\infty.$$

The last theorem of the section shows that τ-modulus of any order $m > 0$ can be used for $1 \leqslant p < \infty$. This is consistent with the previous theorem since $\tau^m(f, t)_\infty = \omega^m(f, t)_\infty$, $\forall t > 0$. This theorem follows from its analogue for almost intertwining approximation (Theorem 14) again.

THEOREM 11. *Let $f \in \mathbf{L}_p[-1, 1] \cap \Delta^0(Y_s)$, $1 \leqslant p < \infty$, and m be a positive integer. Then*

$$E_n^{(0)}(f, \text{alm } Y_s)_p \leqslant C\tau^m(f, n^{-1})_p.$$

If f also belongs to $\mathbf{W}_p^1[-1, 1]$, then

$$E_n^{(0)}(f, \text{alm } Y_s)_p \leqslant Cn^{-1}\omega_\varphi^m(f', n^{-1})_p.$$

6. WEAK INTERTWINING APPROXIMATION

6.1. *Almost Intertwining Approximation*

We first prove the following result for almost intertwining spline approximation. We remind the reader that $J_j^- := [y_j - \Delta_n(y_j), y_j]$ and $J_j^+ := [y_j, y_j + \Delta_n(y_j)]$.

THEOREM 12. *Let* $f \in \mathbf{L}_p[-1, 1]$, $m > 0$, $n > 0$ *and* Y_s *be given. Let* $\mathbf{T}_k := \{t_i\}$ *be a single-knot sequence, with auxiliary knots added as before Lemma* 3, *that has at least* $2(m-1)^2+1$ *knots in the interior of each of* J_j^- *and* J_j^+, $j = 1, ..., s$, *if they do not intersect* J_{j-1}^+ *or* J_{j+1}^-, *respectively. Then there exists an almost intertwining pair of splines* $\{\bar{S}, S\}$ *of order* m *on the knot sequence* $\mathbf{T}_k$ *such that for* $i = 0, ..., k-1$,

$$\|\bar{S} - f\|_{L_p(I_i)} + \|S - f\|_{L_p(I_i)} \leqslant C\tau^m(f, |\mathscr{I}_i|, \mathscr{I}_i)_p,$$

where $I_i := [t_i, t_{i+1}]$, $\mathscr{I}_i$ *is an interval such that* $I_i \subset \mathscr{I}_i \subseteq [t_{i-6(m-1)^2}, t_{i+6(m-1)^2}]$, *and* C *is a constant depending on* m *and the maximum ratios* $\Delta t_i / \Delta t_{i+1}$ *of lengths of neighboring subintervals* I_i *and* I_{i+1}.

Proof. Since the theorem can be easily proved from results of one-sided approximation and Beatson's blending lemma [1, Lemma 3.2] by somehow standard techniques, (see, for example, Lemma 3, Theorem 4 and [8–13]), we only sketch the proof. We first construct overlapping local polynomials of degree $m-1$ by using one-sided approximations. The adjacent local polynomials are then blended by Beatson's Lemma. The error estimate is similar to that of Theorem 4. ∎

From this and Theorem A, we can prove the following two theorems.

THEOREM 13. *Let* $f \in C[-1, 1]$, $m > 0$, *and* Y_s *be given. Then*

$$\tilde{E}_n(f, \text{alm } Y_s)_\infty \leqslant C\omega_\varphi^m(f, n^{-1})_\infty.$$

Moreover, there exists an almost intertwining pair of polynomials $\{P_n, Q_n\}$ *such that*

$$|P_n(x) - f(x)| + |f(x) - Q_n(x)| \leqslant C\omega^m(f, \Delta_n(x))_\infty.$$

Proof. Let $r = 2m+1$ and $T_{k-2s} := \{x_i = \cos(i\pi/k)\}_{i \in I_k(Y_s)}$, where $n \leqslant k \leqslant C(Y_s, r)\, n$ such that T_{k-2s} satisfies the hypothesis in Theorem 12. Then, from Theorem 12, there exists an almost intertwining pair of splines $\{\bar{S}, S\}$ of order r on the knot sequence T_{k-2s} such that

$$\|\bar{S} - f\|_{C(I_i)} + \|S - f\|_{C(I_i)} \leqslant C\omega^r(f, |\mathscr{I}_i|, \mathscr{I}_i)_\infty.$$

Moreover, we have

$$\begin{aligned} E_{r-1}(\bar{S}, I_i \cup I_{i+1})_\infty &\leqslant E_{r-1}(\bar{S} - f, I_i \cup I_{i+1})_\infty + E_{r-1}(f, I_i \cup I_{i+1})_\infty \\ &\leqslant \|\bar{S} - f\|_{C(I_i \cup I_{i+1})} + C\omega^r(f, |I_i \cup I_{i+1}|, I_i \cup I_{i+1})_\infty \\ &\leqslant C\omega^r(f, |\mathscr{I}_i|, \mathscr{I}_i)_\infty, \end{aligned}$$

and, similarly,

$$E_{r-1}(S, I_i \cup I_{i+1})_\infty \leqslant C\omega^r(f, |\mathscr{I}_i|, \mathscr{I}_i)_\infty.$$

Now, by applying Theorem A to $\bar{S}$ and S, respectively, we obtain intertwining pairs of polynomials $\{\bar{P}_1, \bar{P}_2\}$ and $\{P_1, P_2\}$ such that the estimate (5.2) holds for $|\bar{P}_1(x) - \bar{P}_2(x)|$ and $|P_1(x) - P_2(x)|$. Let $x \in I_i$ and $\psi_i(x) := |I_i|/(|x - x_i| + |I_i|)$. Since $|\mathscr{I}_i| \sim |I_i| \sim \Delta_n(x)$ and $\sum \psi_i^2(x) < \infty$, it follows that

$$\begin{aligned}|\bar{P}_1(x) - \bar{P}_2(x)| &\leqslant C(r, \mu, s) \sum_{j=1}^{k-1} E_{r-1}(\bar{S}, I_j \cup I_{j+1})_\infty \,(\psi_j(x))^\mu \\ &\leqslant C(r, \mu, s) \sum_{j=1}^{k-1} \omega^r(f, |\mathscr{I}_j|, \mathscr{I}_j)_\infty \,(\psi_j(x))^\mu\end{aligned}$$

implies

$$|\bar{P}_1(x) - \bar{P}_2(x)| \leqslant C(r, \mu, s)\, \omega_\varphi^m(f, n^{-1})_\infty$$

and

$$|\bar{P}_1(x) - \bar{P}_2(x)| \leqslant C(r, \mu, s)\, \omega^m(f, \Delta_n(x))_\infty.$$

Similar inequalities hold for $|P_1(x) - P_2(x)|$, and therefore, also for $|\bar{P}_1(x) - f(x)|$ and $|f(x) - P_2(x)|$. It is easy to see $\{\bar{P}_1, P_2\}$ is the desired almost intertwining pair of polynomials for f. ∎

The proof of the following theorem is similar and thus will be omitted.

THEOREM 14. *Let $f \in \mathbf{L}_p[-1, 1]$, $1 \leqslant p < \infty$, $m > 0$, and Y_s be given. Then*

$$\tilde{E}_n(f, \text{alm } Y_s)_p \leqslant C\tau^m(f, n^{-1})_p.$$

If f also belong to $\mathbf{W}_p^1[-1, 1]$, then

$$\tilde{E}_n(f, \text{alm } Y_s)_p \leqslant Cn^{-1}\omega_\varphi^m(f', n^{-1})_p.$$

6.2. *Nearly Intertwining Approximation*

In the rest of the paper, we show that the rate of nearly intertwining approximation can not be expressed in terms of τ- and ω-moduli of f, nor in terms of $\|f\|_p$, even if f is infinitely continuously differentiable (Theorem 15), which is no improvement over the ordinary intertwining approximation. This is because, probably, we still require $P - f$ and $f - Q$ change sign *simultaneously* at each $\tilde{y}_j$. If the first derivative of f is used in the bound, however, it has the optimal rate as unconstrained approximation (Theorem 17).

THEOREM 15. *For every $n \in \mathcal{N}$, $0 < p \leqslant \infty$, and $A > 0$, there exists a function $f \in C^\infty[-1, 1]$ such that for every $\tilde{y} \in [-1, 1]$ and every pair of polynomials P and $Q \in \mathbf{P}_n$ with $P - f \in \Delta^0(\tilde{Y}_1)$ and $f - Q \in \Delta^0(\tilde{Y}_1)$, where $\tilde{Y}_1 := \{\tilde{y}\}$, the following inequality holds*:

$$\|P - Q\|_p > A \max\{\|f\|_p, \tau(f, 1)_p\}. \tag{6.1}$$

Proof. Let $n \in \mathcal{N}$, $0 < p \leqslant \infty$, and $A > 0$ be fixed, and define

$$f(x) := \sin(bx),$$

where b is a large positive number to be chosen later. Suppose that $\tilde{y} \in [0, 1]$ (if $\tilde{y} \in [-1, 0)$, then considerations are similar). If $\tilde{y} \in [2k(\pi/b), (2k+1)\pi/b]$ for some $k \in \mathcal{Z}$, we have $P(\tilde{y}) = f(\tilde{y}) \geqslant 0$ and $P((2k - 1/2)\pi/b) \leqslant f((2k - 1/2)\pi/b) = -1$, and, therefore, $\exists \xi \in ((2k - 1/2)\pi/b, \tilde{y})$ such that

$$|P'(\xi)| > \frac{2b}{3\pi}.$$

Similarly, if $\tilde{y} \in ((2k+1)\pi/b, (2k+2)\pi/b]$, then $Q(\tilde{y}) = f(\tilde{y}) \leqslant 0$ and $Q((2k + 1/2)\pi/b) \geqslant f((2k + 1/2)\pi/b) = 1$, and, thus, $\exists \xi \in ((2k + 1/2)\pi/b, \tilde{y})$ such that

$$|Q'(\xi)| > \frac{2b}{3\pi}.$$

This implies that

$$\|P'\|_\infty + \|Q'\|_\infty \geqslant \frac{2b}{3\pi}.$$

Suppose now that P and Q satisfy the inequality

$$\|P - Q\|_p \leqslant A \max\{\|f\|_p, \tau(f, 1)_p\}.$$

Taking into account that $\|f\|_p \leqslant 2^{1/p}$ and $\tau(f, 1)_p \leqslant 2^{1+1/p}$ we have the following estimates

$$\begin{aligned} \frac{2b}{3\pi} &\leqslant \|P'\|_\infty + \|Q'\|_\infty \leqslant n^2(\|P\|_\infty + \|Q\|_\infty) \\ &\leqslant M(\|P\|_p + \|Q\|_p) \leqslant M(\|P - f\|_p + \|Q - f\|_p + 2\|f\|_p) \\ &\leqslant 2M(\|P - Q\|_p + \|f\|_p) \leqslant 2^{2+1/p} M(A+1), \end{aligned}$$

where the constant M depends on n and p but not on b. Choosing $b > 3 \cdot 2^{1+1/p} \pi M(A+1)$ gives the desired contradiction. ∎

The key to the proof of Theorem 17 is the lemma below. The proof of the theorem itself is then somehow standard (see the proofs of Theorems 4 and 12) and thus will be omitted.

LEMMA 16. *Let $f \in \mathbf{W}_p^1[-1, 1]$, and let $I := [a, b]$ and $\tilde{I} \subseteq I$ be two subintervals of $[-1, 1]$. Then there exist $y_1 \in \tilde{I}$ and two polynomials p_1 and p_2 of degree $m \geqslant 1$ such that $\{p_1, p_2\}$ is an intertwining pair for f on I with respect to $Y_1 := \{y_1\}$ that satisfies*

$$\|p_1 - p_2\|_{L_p(I)} \leqslant C\,|I|\,\omega^m(f', |I|, I)_p, \tag{6.2}$$

where C depends on m and the ratio $|I|/|\tilde{I}|$.

Proof. It is well-known that for any integrable function g on $(-\infty, \infty)$,

$$|\{x\colon Mg(x) > t\}| \leqslant ct^{-1} \int_{-\infty}^{\infty} |g|, \qquad t > 0,$$

where $Mg(x) := \sup_{J \ni x} |J|^{-1} \int_J |g|$ is the Hardy–Littlewood maximal operator. Let

$$F(x) := \begin{cases} f(a), & x < a \\ f(x), & x \in [a, b] \\ f(b), & x > b, \end{cases}$$

and $t = 2c\,|\tilde{I}|^{-1} \int_I |f'|$, and apply the maximal operator to F', we have

$$|\{x\colon M(F')(x) > t\}| \leqslant ct^{-1} \int_{-\infty}^{\infty} |F'| = ct^{-1} \int_I |F'| = ct^{-1} \int_I |f'| = \tfrac{1}{2}|\tilde{I}|.$$

Thus there exists $y_1 \in \tilde{I}$ such that $M(F')(y_1) \leqslant t$. Let $l_1(x) = f(y_1) + t(x - y_1)$ and $l_2(x) = f(y_1) - t(x - y_1)$, then they form a intertwining pair of f on I with respect to $Y_1 = \{y_1\}$. This is because we have, for $l_1(x)$, $y_1 \leqslant x \leqslant b$, for example,

$$\begin{aligned} l_1(x) &= f(y_1) + t(x - y_1) \geqslant f(y_1) + (x - y_1)\, M(F')(y_1) \\ &\geqslant f(y_1) + \int_{y_1}^{x} |F'| = f(y_1) + \int_{y_1}^{x} |f'| \geqslant f(x). \end{aligned}$$

For an error estimate of l_1 and l_2, we first note for $\forall x \in I$,

$$|l_1(x) - l_2(x)| = 2t\,|x - y_1| = 4c\,|\tilde{I}|^{-1}\,|x - y_1| \int_I |f'| \leqslant C \int_I |f'|$$
$$\leqslant C\,|I|^{1/q}\,\|f'\|_{L_p(I)},$$

where $1/p + 1/q = 1$. It follows that

$$\|l_1 - l_2\|_{L_p(I)} \leqslant C\,|I|\,\|f'\|_{L_p(I)}. \tag{6.3}$$

Let P' be a best polynomial approximation to f' on I of degree $m-1$, and $P := \int_a^x P'(t)\,dt$. To prove (6.2), we apply (6.3) to $f - P$, then $\{l_1, l_2\}$ is an intertwining pair for $f - P$. Define $p_i := l_i + P$, $i = 1, 2$. Obviously $\{p_1, p_2\}$ is an intertwining pair of polynomials of degree m for f on I with

$$\|p_1 - p_2\|_{L_p(I)} = \|l_1 - l_2\|_{L_p(I)} \leqslant C\,|I|\,\|f' - P'\|_{L_p(I)} \leqslant C\,|I|\,\omega^m(f', |I|, I)_p,$$

where Whitney's Theorem has been used. ∎

THEOREM 17. *Suppose $f \in \mathbf{W}_p^1[-1, 1]$, $k > 0$ and $n > m > 0$.*

(i) *If $\mathbf{T}_k$ is a single-knot sequence, with auxiliary knots added as before Lemma 3, that has at least $2(r-1)^2 + 1$ knots in each of (y_{j-1}, y_j), $j = 2, ..., s$, then there exists a nearly intertwining pair $\{S_1, S_2\}$ of splines of order $m+1$ on $\mathbf{T}_k$ for f with respect to Y_s satisfying*

$$\|S_1 - S_2\|_p \leqslant C\,\Delta\mathbf{T}_k\,\omega^m(f', \Delta\mathbf{T}_k)_p, \tag{6.4}$$

where C depends on m and on ratios $\Delta t_i/\Delta t_{i+1}$ of lengths of neighboring subintervals I_i and I_{i+1}. It also depends on ratios $\Delta t_i/\Delta_n(y_j)$ if $y_j \in I_i$ and $\Delta_n(y_j) \ll \Delta t_i = |I_i|$.

(ii) *There exists a nearly intertwining pair $\{P_1, P_2\}$ of polynomials of degree $C_1 n$ for f with respect to Y_2 satisfying*

$$\|P_1 - P_2\|_p \leqslant C n^{-1} \omega_\varphi^m(f', n^{-1})_p, \tag{6.5}$$

where C_1 depends only on m while C depends on m and Y_s.

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